On M-stationary points for a stochastic equilibrium problem under equilibrium constraints in electricity spot market modeling*

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This paper is dedicated to our friend and colleague Jiri Outrata on the occasion of his 60th birthday

Abstract

Modeling several competitive leaders and followers acting in an electricity market leads to coupled systems of mathematical programs with equilibrium constraints, called *equilibrium problems with equilibrium constraints (EPECs)*. We consider a simplified model for competition in electricity markets under uncertainty of demand in an electricity network as a (stochastic) multi-leader-follower game. First order necessary conditions are developed for the corresponding stochastic EPEC based on a result of Outrata [17]. For applying the general result an explicit representation of the co-derivative of the normal cone mapping to a polyhedron is derived (Proposition 3.2). Later the co-derivative formula is used for verifying constraint qualifications and for identifying M-stationary solutions of the stochastic EPEC if the demand is represented by a finite number of scenarios.

Keywords: Electricity markets, bidding, noncooperative games, equilibrium constraint, EPEC, optimality condition, co-derivative, random demand.

1 Introduction

In [17], J. Outrata formulated first order necessary conditions for the following equilibrium problem with equilibrium constraints (EPEC):

$$\min\left\{f_i\left(x^i,z\right)\left|0\in F(x,z)+N_U(z)\right.\right\}\quad (i=1,\dots,N) \tag{EPEC}$$

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Here, the $x^i \in \mathbb{R}^n$ refer to decisions taken by N players (e.g., market competitors), whose objective functions f_i do not only depend on their own decisions x^i but also on some parameter z which might represent an exterior decision (e.g., in a *leader-follower* system). All decisions together are linked by a generalized equation $0 \in F(x,z)+N_U(z)$ which could model some equilibrium constraint or the solution of a parameter-dependent optimization problem. It is assumed, that U is some closed convex set and N_U refers to its normal cone. In principle, (EPEC) is nothing else but a coupled system of mathematical programs with equilibrium constraints (MPECs), where each single MPEC describes the optimization problem solved by the individual players given the decision of the other players. The vector $\left(\bar{x}^1,\ldots,\bar{x}^N,\bar{z}\right)$ is declared to be a solution to (EPEC), if for $i=1,\ldots,N$ the vectors $(\bar{x}^i,\bar{z})$ are solutions to the MPEC

$$\min\left\{f_i\left(y,z\right)\left|0\in F(\bar{x}^1,\ldots,\bar{x}^{i-1},y,\bar{x}^{i+1}\bar{x}^N,\bar{z})+N_U(\bar{z})\right.\right\},$$

i.e., non of the players can improve his decision given the decisions of his competitors. As pointed out in [17], these MPECs are typically nonconvex even under convexity assumptions on the data f_i, F, U. Therefore it makes sense to identify possible solutions by means of first order necessary conditions. In [17], it was proposed to do so by using Mordukhovich's co-derivative D^* of multifunctions (see [15]) as a basic tool. For recent extensions of these ideas (e.g., to stability issues in the context of quasi-variational inequalities), we refer to [16] (see also [15]). We cite the following Theorem from [17], slightly adapted to the purposes of our paper:

Theorem 1.1 *Let $(\bar{x},\bar{z})$ be a solution to (EPEC). If, for all $i=1,\ldots,N$, the multifunctions*

$$u\mapsto\left\{\left(x^i,z\right)\left|u\in F(\bar{x}^1,\ldots,\bar{x}^{i-1},x^i,\bar{x}^{i+1},\ldots,\bar{x}^N,z)+N_U(z)\right.\right\}$$

are polyhedral or satisfy the constraint qualification

$$\left.\begin{array}{l}0=\left(\nabla_{x^i}F\left(\bar{x},\bar{z}\right)\right)^T v\\ 0\in\left(\nabla_z F\left(\bar{x},\bar{z}\right)\right)^T v+D^*N_U(\bar{z},-F\left(\bar{x},\bar{z}\right))(v)\end{array}\right\}\Longrightarrow v=0,$$

then, for all $i=1,\ldots,N$, there exist $\bar{v}^i$ such that

$$\begin{aligned}0 &= \nabla_{x^i}f_i\left(\bar{x},\bar{z}\right)+\left(\nabla_{x^i}F\left(\bar{x},\bar{z}\right)\right)^T\bar{v}^i &(1)\\ 0 &\in \nabla_z f_i\left(\bar{x},\bar{z}\right)+\left(\nabla_z F\left(\bar{x},\bar{z}\right)\right)^T\bar{v}^i+D^*N_U(\bar{z},-F\left(\bar{x},\bar{z}\right))(\bar{v}^i). &(2)\end{aligned}$$

We shall adopt from [17] the name M(ordukhovich)- *stationary point* for any $(\bar{x},\bar{z})$ satisfying (1) and (2). The main difficulty in the verification of both the constraint qualification and the optimality conditions (1) and (2) is the computation of the co-derivative D^*N_U to the normal cone mapping associated with U. Explicit formulae ready to use can be found in [2] and [18] for the cases of U being a nonnegative orthant or a rectangle. On the other hand, many practical applications like electricity spot market modeling lead to sets U which are general polyhedra. The purpose of this note is threefold: first, it is intended to apply the ideas presented so far to a simplified model of electricity markets under an independent system operator regime similar to [4] and [11]. Second, and subordinate to this aim, an explicit formula for D^*N_U is derived for general polyhedra U. Third, the whole problem is put into a stochastic framework which is of much interest due to uncertainties in electricity demands. For discrete distributions, a characterizing system of relations for identifying M-stationary solutions is provided and such solutions are explicitly calculated for a simple example.

Since electricity production and trading decisions of smaller power firms (followers) do not influence market prices, electricity portfolio optimization models for such firms may be developed without regarding their market interactions. Inputs of portfolio optimization models are stochastic price and demand processes in the relevant time horizon (see, e.g., [3]). To extend stochastic portfolio optimization models to firms having market power (leaders), the use of modified market prices is suggested, e.g., in [1].

To investigate the behavior of power firms in deregulated electricity markets, game-theoretic models are employed (see, e.g., [7, 8, 28]). Such models have to incorporate the specific features of electricity markets, namely, the transmission network and the bidding of price-quantity pairs of each generator in the network. When modeling single-leader-follower games one arrives at mathematical programs with equilibrium constraints (MPECs). Presently, theory and numerical methods for MPECs is well developed. We refer to the monographs [14, 19, 5], the survey [12] and to [25, 6]. Extensions to stochastic MPECs (SMPECs) can be found in [26, 27] and applications to electricity markets are discussed, e.g., in [9, 21].

The modeling of multi-leader-follower games leads to coupled systems of MPECs or equilibrium problems with equilibrium constraints (EPECs). In recent years, much effort has been directed to the theory of such games [20] and to numerical methods [13] based on nonlinear programming and nonlinear complementarity (re)formulations. Furthermore, EPEC models for electricity markets with generators and customers located on a network have been developed and analyzed in [11, 10, 22]. A stochastic EPEC (SEPEC) modeling an electricity market under demand uncertainty is studied in [4].

2 A simplified model for competition in electricity spot markets

In the following, we consider a model for competition in electricity spot markets which is a simplified for the purpose of our analysis version of models presented in [4] and [11]. We assume that some electricity network is represented by an oriented graph, whose m edges correspond to transmission lines and whose N nodes refer to places at which a demand for electricity is observed and at which electricity is generated. Neglecting, for the sake of simplicity, transmission losses, the satisfaction of demand may be modeled as

$$q + By \geq d. \tag{3}$$

Here, $d \in \mathbb{R}^N$ refers to the vector of demands at each node, $q \in \mathbb{R}^N$ is the vector of electricity generated at the same nodes and $y \in \mathbb{R}^m$ represents the oriented flow vector of electricity along the edges of the graph. B is the incidence matrix of the electricity network. Typically, q and y are simply bounded by

$$0 \leq q \leq \hat{q}, \quad -\hat{y} \leq y \leq \hat{y},$$

where the inequality signs are to be understood component-wise. Generators bid a cost function to an independent system operator (ISO):

$$c_i(q_i) = \alpha_i q_i + \beta_i q_i^2 \quad (i = 1, \ldots N).$$

These may differ from the true cost functions

$$C_i(q_i) = \gamma_i q_i + \delta_i q_i^2 \quad (i = 1, \ldots N).$$

Throughout the paper, we shall assume that $\beta_i > 0$ for $i = 1, \ldots, N$, thus accepting the idea that cost functions are typically convex and leaving aside the purely linear case. More general cost functions were allowed in [4]. Here, we restrict ourselves to the quadratic case as considered in [11]. The ISO determines a vector of generated electricity satisfying the constraints above and minimizing the overall costs:

$$\min_{q,y} \left\{ \sum_{i=1}^{N} c_i(q_i) \,|(q,y) \in G \right\}, \tag{4}$$

where

$$G := \left\{ (q,y) \in \mathbb{R}^{N+m} \middle| \, q + By \geq d, \; 0 \leq q \leq \hat{q}, \; -\hat{y} \leq y \leq \hat{y} \right\}.$$

Note that, by convexity, an optimal solution q^* of (4) is characterized as a solution to the generalized equation

$$0 \in \begin{pmatrix} \alpha + 2\,[\operatorname{diag} \beta]\, q \\ 0 \end{pmatrix} + N_G(q,y). \tag{5}$$

Here, $[\operatorname{diag} \beta]$ denotes the diagonal matrix composed of diagonal entries β_i. With q^* being an optimal solution to (4), the clearing price charged by generator i amounts to the derivative of its bid cost function at q_i^* (see [11]):

$$\pi_i = \alpha_i + 2\beta_i q_i^*.$$

Thus, generator i's profit calculates as

$$(\alpha_i - \gamma_i)\, q_i^* + (2\beta_i - \delta_i)\, (q_i^*)^2.$$

Therefore, given some fixed bid coefficients $\left(\bar{\alpha}_j, \bar{\beta}_j\right)$ of the remaining competitors $j \neq i$, generator i solves the following mathematical program with equilibrium constraints (MPEC):

$$\max_{\alpha_i, \beta_i, q, y} \left\{ (\alpha_i - \gamma_i)\, q_i + (2\beta_i - \delta_i)\, q_i^2 \middle| 0 \in \begin{pmatrix} \theta(\alpha_i, \beta_i, q) \\ 0 \end{pmatrix} + N_G(q,y) \right\}, \tag{6}$$

where

$$\theta(\alpha_i, \beta_i, q) := (\bar{\alpha}_1, \ldots, \bar{\alpha}_{i-1}, \alpha_i, \bar{\alpha}_{i+1}, \ldots, \bar{\alpha}_N) + 2\left[\operatorname{diag}\left(\bar{\beta}_1, \ldots, \bar{\beta}_{i-1}, \beta_i, \bar{\beta}_{i+1}, \ldots, \bar{\beta}_N\right)\right] q$$

(compare (5)). Since all competitors solve a similar MPEC given the decisions of the remaining ones, the coupled system of MPECs

$$\min_{\alpha_i, \beta_i, q, y} \left\{ (\gamma_i - \alpha_i)\, q_i + (\delta_i - 2\beta_i)\, q_i^2 \,\middle|\, 0 \in \begin{pmatrix} \alpha + 2\,[\operatorname{diag} \beta]\, q \\ 0 \end{pmatrix} + N_G(q, y) \right\} \quad (i = 1, \ldots, N) \tag{7}$$

is called an EPEC (equilibrium problem with equilibrium constraints). This EPEC falls into the general class of type (EPEC) presented in the introduction. Indeed, in our specific model, one has to put $x^i := (\alpha_i, \beta_i)$, $z := (q, y)$, $U := G$ as well as

$$\begin{aligned} f_i\,(\alpha_i, \beta_i, q, y) &= (\gamma_i - \alpha_i)\, q_i + (\delta_i - 2\beta_i)\, q_i^2 \\ F\,(\alpha, \beta, q, y) &= \begin{pmatrix} \alpha + 2\,[\operatorname{diag} \beta]\, q \\ 0 \end{pmatrix}. \end{aligned} \tag{8}$$

Specializing Theorem 1.1 from the introduction to our setting, we obtain:

Theorem 2.1 *Let $\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)$ be a solution to (7). If, for all $i = 1, \ldots, N$, the multifunctions*

$$\begin{aligned} u \;\mapsto\; & \left\{ (\alpha_i, \beta_i, q, y) \,\middle|\, u \in F(\bar{\alpha}_1, \bar{\beta}_1, \ldots, \bar{\alpha}_{i-1}, \bar{\beta}_{i-1}, \alpha_i, \beta_i, \bar{\alpha}_{i+1}, \bar{\beta}_{i+1}, \ldots, \bar{\alpha}_N, \bar{\beta}_N, q, y) \right. \\ & \left. + N_G(q, y) \right\} \end{aligned} \tag{9}$$

are polyhedral or satisfy the constraint qualification

$$\left. \begin{aligned} & 0 = \left(\nabla_{(\alpha_i, \beta_i)} F\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)\right)^T v \\ & 0 \in \left(\nabla_{(q,y)} F\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)\right)^T v + D^* N_G((\bar{q}, \bar{y}), -F\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right))(v) \end{aligned} \right\} \Longrightarrow v = 0, \tag{10}$$

then, for all $i = 1, \ldots, N$, there exist $\bar{v}^i$ such that

$$0 = \nabla_{(\alpha_i, \beta_i)} f_i\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right) + \left(\nabla_{(\alpha_i, \beta_i)} F\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)\right)^T \bar{v}^i \tag{11}$$

$$\begin{aligned} 0 \in\; & \nabla_{(\alpha_i, \beta_i)} f_i\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right) + \left(\nabla_{(\alpha_i, \beta_i)} F\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)\right)^T \bar{v}^i \\ & + D^* N_G(\bar{q}, \bar{y}, -F\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right))(\bar{v}^i). \end{aligned} \tag{12}$$

One observes that the difficult part both in the verification of the constraint qualification and in the application of the first order necessary condition consists in calculating the co-derivative $D^* N_G$. This is the aim of the following section.

3 On the co-derivative of the normal cone mapping to a polyhedron

This section is devoted to the derivation of an explicit formula for the co-derivative of the normal cone mapping to a polyhedron. Before addressing this topic, we recall the definition of the Mordukhovich normal cone (also called limiting normal cone) and the the induced co-derivative (see [15]):

Definition 3.1 *Let $S \subseteq \mathbb{R}^n$ be an arbitrary set and $\bar{x} \in \operatorname{cl} S$. Then, the Mordukhovich normal cone to S at $\bar{x}$ is defined by*

$$N_S(\bar{x}) := \operatorname{Limsup}_{x \to \bar{x}, x \in S} [T_S(x)]^*,$$

where $[T_S(x)]^*$ refers to the negative polar of the contingent cone $T_S(x)$ to S at x and 'Limsup' denotes the upper limit in the sense of Kuratowski-Painlevé convergence.

For a multifunction $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$, consider a point of its graph: $(x, y) \in \operatorname{gph} \Phi$. The Mordukhovich normal cone induces the following co-derivative $D^*\Phi(x, y) : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ of Φ at (x, y):

$$D^*\Phi(x, y)(y^*) = \{x^* \in \mathbb{R}^n | (x^*, -y^*) \in N_{\operatorname{gph} \Phi}(x, y)\} \quad \forall y^* \in \mathbb{R}^p.$$

Now, we consider a polyhedron $C := \{x \in \mathbb{R}^n | Ax \leq b\}$, where $b \in \mathbb{R}^m$ and A is a matrix of order (m, n). Let $(x^0, v^0) \in \operatorname{gph} N_C$. As C is convex, the Mordukhovich normal cone N_C reduces to the normal cone in the sense of convex analysis here. In particular $x^0 \in C$ and $v^0 \in N_C(x^0)$. With a_i and b_i referring to the rows of A and components of b, respectively, let

$$I := \{i \in \{1, \ldots, m\} | \langle a_i, x^0 \rangle = b_i\}$$

be the set of active indices at x^0. Since $v^0 \in N_C(x^0)$, there exits $\lambda_i \geq 0$ for $i \in I$, such that

$$v^0 = \sum_{i \in I} \lambda_i a_i. \tag{13}$$

We introduce the following subset of I:

$$J := \{i \in I | \lambda_i > 0\}.$$

Finally, for each index subset $I' \subseteq I$, we introduce the closed cone

$$F_{I'} = \{h \in \mathbb{R}^n | \langle a_i, h \rangle \leq 0 \quad (i \in I \backslash I')\}, \quad \langle a_i, h \rangle = 0 \quad (i \in I')\} \tag{14}$$

as well as the characteristic index set

$$\chi(I') := \{j \in I | \langle a_j, h \rangle = 0 \quad \forall h \in F_{I'}\}. \tag{15}$$

Proposition 3.2 *With the notation introduced above, one has that*

$$N_{\mathrm{gph}\, N_C}\left(x^0, v^0\right) = \bigcup_{J \subseteq I_1 \subseteq I_2 \subseteq I} P_{I_1,I_2} \times Q_{I_1,I_2},$$

where

$$\begin{aligned} P_{I_1,I_2} &= \operatorname{con}\{a_i | i \in \chi(I_2) \setminus I_1\} + \operatorname{span}\{a_i | i \in I_1\} \\ Q_{I_1,I_2} &= \{h \in \mathbb{R}^n | \langle a_i, h\rangle = 0 \quad (i \in I_1), \quad \langle a_i, h\rangle \le 0 \quad (i \in \chi(I_2) \setminus I_1)\}. \end{aligned}$$

Here, con *and* span *refer to the convex conic and linear hull, respectively.*

Proof. First note, that the set $\mathrm{gph}\, N_C$ is no longer convex although the polyhedron C is so. As a consequence, the Mordukhovich normal cone $N_{\mathrm{gph}\, N_C}(x^0, v^0)$ to this set evaluated at the point (x^0, v^0) needs not be convex either. According to a well-known result by Dontchev and Rockafellar ([2, Proof of Theorem 2]), one has that

$$N_{\mathrm{gph}\, N_C}\left(x^0, v^0\right) = \bigcup_{F_j \subseteq F_i} (F_i - F_j)^* \times (F_i - F_j), \tag{16}$$

where the F_i are the closed faces of the cone

$$K^0 := T_C\left(x^0\right) \cap \{v^0\}^{\perp}$$

and T_C denotes the tangent cone to C in the sense of convex analysis. As in Definition 3.1, we use an asterisk for denoting the negative polar (or dual) cone. Combining the well-known representation

$$T_C\left(x^0\right) = \{h \in \mathbb{R}^n | \langle a_i, h\rangle \le 0 \quad (i \in I)\},$$

with (13) and the definition of the index set J, one immediately derives that

$$K^0 = \{h \in \mathbb{R}^n | \langle a_i, h\rangle \le 0 \quad (i \in I \setminus J), \quad \langle a_i, h\rangle = 0 \quad (i \in J)\}.$$

Now, any closed face of K^0 is given by a cone $F_{I'}$ as introduced in (14) and with I' being an arbitrary index set with $J \subseteq I' \subseteq I$. Clearly, the implication

$$I_1 \subseteq I_2 \Longrightarrow F_{I_2} \subseteq F_{I_1}$$

holds true for all index sets I_1, I_2 such that $J \subseteq I_1, I_2 \subseteq I$. While the reverse implication cannot be derived in general, one may easily show the following for the same index sets:

$$F_{I_2} \subseteq F_{I_1} \Longrightarrow F_{I_2} = F_{I_1 \cup I_2}.$$

In other words, there exists an index set I_3, such that $F_{I_2} = F_{I_3} \subseteq F_{I_1}$ and $I_1 \subseteq I_3$. Summarizing, any pair of index sets I_1, I_2 with $J \subseteq I_1 \subseteq I_2 \subseteq I$ induces a pair of closed faces of K^0 such that one is a subset of the other, and, conversely, any such pair of closed faces of K^0 can be represented by a pair of index sets I_1, I_2 with $J \subseteq I_1 \subseteq I_2 \subseteq I$. Consequently, we may rewrite (16) as

$$N_{\operatorname{gph} N_C}\left(x^0, v^0\right) = \bigcup_{J \subseteq I_1 \subseteq I_2 \subseteq I} \left(F_{I_1} - F_{I_2}\right)^* \times \left(F_{I_1} - F_{I_2}\right). \tag{17}$$

We claim that

$$F_{I_1} - F_{I_2} = Q_{I_1,I_2} \quad \forall I_1, I_2 : J \subseteq I_1 \subseteq I_2 \subseteq I, \tag{18}$$

where Q_{I_1,I_2} is defined in the statement of the proposition. Recall that, by the very definition of χ in (15), one always has that $I_2 \subseteq \chi(I_2) \subseteq I$. Now, given any $h \in F_{I_1} - F_{I_2}$, one has $h = h_1 - h_2$ for some $h_1 \in F_{I_1}$ and $h_2 \in F_{I_2}$. The inclusion $I_1 \subseteq I_2$ along with (14) then implies that

$$\langle a_i, h_1 \rangle = \langle a_i, h_2 \rangle = 0 \quad (i \in I_1); \quad \langle a_i, h_1 \rangle \leq 0 \quad (i \in I \backslash I_1) \quad \langle a_i, h_2 \rangle = 0 \quad (i \in I_2).$$

By (15), we have that $\langle a_i, h_2 \rangle = 0$ for all $i \in \chi(I_2)$. Moreover, $\langle a_i, h_1 \rangle \leq 0$ for all $i \in \chi(I_2) \backslash I_1$. Altogether, this establishes the inclusion '$\subseteq$' of (18).

For the reverse inclusion, let $h \in Q_{I_1,I_2}$ be arbitrary. In case that $\chi(I_2) = I$, it follows form the definition of Q_{I_1,I_2} that $h \in F_{I_1} \subseteq F_{I_1} - F_{I_2}$ (due to $0 \in F_{I_2}$). Hence, we may assume now that $\chi(I_2) \subsetneqq I$. By (15), we have

$$\chi(I_2) = \{j \in I | \langle a_j, h' \rangle = 0 \quad \forall h' \in F_{I_2}\}.$$

As a consequence, for all $j \in I \backslash \chi(I_2)$ there exists some $h_j \in F_{I_2}$ such that $\langle a_j, h_j \rangle < 0$. We put

$$h^* := \sum_{j \in I \backslash \chi(I_2)} h_j.$$

Note that h^* is well-defined by $I \backslash \chi(I_2) \neq \emptyset$. Clearly, $h^* \in F_{I_2}$ and

$$\langle a_i, h^* \rangle = \langle a_i, h_i \rangle + \sum_{\substack{j \in I \backslash \chi(I_2) \\ j \neq i}} \langle a_i, h_j \rangle < 0$$

by definition of h_i and by $\langle a_i, h_j \rangle \leq 0$ for all $j \in I \backslash \chi(I_2)$ (recall that $h_j \in F_{I_2}$). This allows to define

$$t := \max\left\{0, \max_{i \in I \backslash \chi(I_2)} \left\{-\frac{\langle a_i, h \rangle}{\langle a_i, h^* \rangle}\right\}\right\} \geq 0.$$

Finally, put $\bar{h} := h + th^*$. Due to $h \in Q_{I_1,I_2}$ and $h^* \in F_{I_2}$, we have that

$$\langle a_i, h \rangle = 0 \quad (i \in I_1); \quad \langle a_i, h^* \rangle = 0 \quad (i \in \chi(I_2)); \quad \langle a_i, h \rangle \leq 0 \quad (i \in \chi(I_2) \backslash I_1).$$

Consequently, recalling that $I_1 \subseteq I_2 \subseteq \chi(I_2)$, it follows that $\langle a_i, \bar{h} \rangle = 0$ for all $i \in I_1$ and $\langle a_i, \bar{h} \rangle \leq 0$ for all $i \in \chi(I_2) \backslash I_1$. We claim that

$$\langle a_i, \bar{h} \rangle = \langle a_i, h \rangle + t \langle a_i, h^* \rangle \leq 0 \quad \forall i \in I \backslash \chi(I_2).$$

Indeed, the inequality is obvious if $\langle a_i, h\rangle \le 0$, because of $t \ge 0$ and $\langle a_i, h^*\rangle < 0$. If $\langle a_i, h\rangle > 0$, then the same inequality follows from

$$t \ge -\frac{\langle a_i, h\rangle}{\langle a_i, h^*\rangle}$$

by definition of t. Summarizing the previous relations, one arrives at $\bar{h} \in F_{I_1}$. Therefore, $h = \bar{h} - th^* \in F_{I_1} - F_{I_2}$, where we used that $th^* \in F_{I_2}$ due to $t \ge 0$. This finishes the proof of (18).

Evidently, $P_{I_1,I_2} = Q^*_{I_1,I_2}$ for P_{I_1,I_2} as defined in the statement of the proposition. Consequently, the proposition is proved upon referring to (18) and (17). ■

Remark 3.3 *If, the vectors $\{a_i\,|i \in I\}$ happen to be linearly independent, then $\chi(I') = I'$ for all $I' \subseteq I$ and the definitions of P_{I_1,I_2} and Q_{I_1,I_2} in Proposition 3.2 simplify accordingly.*

Corollary 3.4 *In the setting of Proposition 3.2, one has the following:*

$$\begin{aligned} D^*N_C\left(x^0, v^0\right)(s) &\subseteq \operatorname{con}\left\{a_i | i \in \chi\left(I^a(s) \cup I^b(s)\right) \backslash I^a(s)\right\} + \operatorname{span}\left\{a_i | i \in I^a(s)\right\} \\ & \qquad \textit{if } \langle a_i, s\rangle = 0 \quad \forall i \in J \quad \textit{and} \quad \langle a_i, s\rangle \ge 0 \quad \forall i \in \chi(J)\backslash J \\ &\textit{and} \\ D^*N_C\left(x^0, v^0\right)(s) &= \varnothing \quad \textit{otherwise.} \end{aligned}$$

Here,

$$I^a(s) := \{i \in I |\, \langle a_i, s\rangle = 0\}, \quad I^b(s) := \{i \in I |\, \langle a_i, s\rangle > 0\}.$$

Proof. ¿From the definition of the co-derivative and from Proposition 3.2, it follows that

$$\begin{aligned} D^*N_C\left(x^0, v^0\right)(s) &= \left\{r |\, (r, -s) \in N_{\operatorname{gph} N_C}\left(x^0, v^0\right)\right\} \\ &= \{r | \exists I_1, I_2 : J \subseteq I_1 \subseteq I_2 \subseteq I, r \in P_{I_1,I_2}, -s \in Q_{I_1,I_2}\}. \qquad (19) \end{aligned}$$

Since $Q_{I_1,I_2} \subseteq Q_{J,J}$ for all I_1, I_2 with $J \subseteq I_1 \subseteq I_2 \subseteq I$, it follows that $D^*N_C\left(x^0, v^0\right)(s)$ is non-empty only if $-s \in Q_{J,J}$ which means, by definition, that $\langle a_i, s\rangle = 0$ for all $i \in J$ and $\langle a_i, s\rangle \ge 0$ for all $i \in \chi(J)\backslash J$. This proves the second statement of the corollary. We show that

$$Q_{I^a(s), I^a(s)\cup I^b(s)} \subseteq Q_{I_1,I_2} \quad \forall I_1, I_2 : J \subseteq I_1 \subseteq I_2 \subseteq I \quad \forall s : -s \in Q_{I_1,I_2}. \qquad (20)$$

Indeed, the definitions of the respective index sets yield that $I_1 \subseteq I^a(s)$ and

$$\chi(I_2) \subseteq I^a(s) \cup I^b(s) \subseteq \chi(I^a(s) \cup I^b(s)).$$

Now, if $h \in Q_{I^a(s), I^a(s)\cup I^b(s)}$, then

$$\langle a_i, h\rangle = 0 \quad \forall i \in I^a(s), \quad \langle a_i, h\rangle \le 0 \quad \forall i \in \chi\left(I^a(s) \cup I^b(s)\right) \backslash I^a(s).$$

It follows that

$$\langle a_i, h\rangle = 0 \quad \forall i \in I_1, \quad \langle a_i, h\rangle \leq 0 \quad \forall i \in \chi(I_2)\backslash I^a(s).$$

Due to

$$\chi(I_2)\backslash I_1 \subseteq (\chi(I_2)\backslash I^a(s)) \cup (I^a(s)\backslash I_1)\,,$$

one arrives that $\langle a_i, h\rangle \leq 0 \quad \forall i \in \chi(I_2)\backslash I_1$, whence $h \in Q_{I_1,I_2}$. This establishes (20). Recalling that $P_{I_1,I_2} = Q^*_{I_1,I_2}$, it results from (20) that

$$P_{I_1,I_2} = Q^*_{I_1,I_2} \subseteq Q^*_{I^a(s),I^a(s)\cup I^b(s)} = P_{I^a(s),I^a(s)\cup I^b(s)}.$$

Now, we may continue (19) as

$$D^*N_C\left(x^0, v^0\right)(s) \subseteq P_{I^a(s),I^a(s)\cup I^b(s)},$$

which proves the first statement of the corollary. ■

The following simplification of Corollary 3.4 is possible under the assumption of linear independence:

Corollary 3.5 *If the $\{a_i\,|i \in I\}$ are linearly independent, then Corollary 3.4 simplifies to*

$$\begin{aligned} D^*N_C\left(x^0, v^0\right)(s) &= \operatorname{con}\{a_i|i \in I^b(s)\} + \operatorname{span}\{a_i|i \in I^a(s)\} \\ &\quad \textit{if } \langle a_i, s\rangle = 0 \quad \forall i \in J, \\ \textit{and} & \\ D^*N_C\left(x^0, v^0\right)(s) &= \varnothing \quad \textit{otherwise.} \end{aligned}$$

Proof. In view of Remark 3.3, we have that $\chi(J) = J$ and, by $I^a(s) \cap I^b(s) = \varnothing$, that

$$\chi\left(I^a(s) \cup I^b(s)\right)\backslash I^a(s) = \left(I^a(s) \cup I^b(s)\right)\backslash I^a(s) = I^b(s). \tag{21}$$

Then, Corollary 3.4 yields the assertion of the proposition with the first identity replaced by an inclusion. To prove the reverse inclusion, let

$$r \in \operatorname{con}\{a_i|i \in I^b(s)\} + \operatorname{span}\{a_i|i \in I^a(s)\}$$

be arbitrary. Then, by definition and due to (21), $r \in P_{I^a(s),I^a(s)\cup I^b(s)}$. Exploiting (21) once more, the definitions of $I^a(s)$ and $I^b(s)$ provide that $-s \in Q_{I^a(s),I^a(s)\cup I^b(s)}$. Consequently, $r \in D^*N_C\left(x^0, v^0\right)(s)$ by definition of D^*N_C. This finishes the proof. ■

Another simplification of Corollary 3.4 can be obtained without linear independence, but under the assumption of strict complementarity (i.e., $\lambda_i > 0$ for all $i \in I$ in (13)):

Corollary 3.6 *If $J = I$, then*

$$D^*N_C\left(x^0, v^0\right)(s) = \begin{cases} \operatorname{span}\{a_i | i \in I\} & \text{if } \langle a_i, s\rangle = 0 \quad \forall i \in I \\ \varnothing & \text{otherwise} \end{cases}.$$

Proof. The second case follows immediately from Corollary 3.4 and from $J = I$. Now, in the first case, one has $\langle a_i, s\rangle = 0$ for all $i \in J$, hence $J \subseteq I^a(s) \subseteq I$. Consequently, $I^a(s) = I$ and $I^b(s) = \varnothing$. Then,

$$D^*N_C\left(x^0, v^0\right)(s) \subseteq \operatorname{span}\{a_i | i \in I\}$$

by virtue of Corollary 3.4. For the reverse inclusion, let $r \in \operatorname{span}\{a_i | i \in I\}$ be arbitrary. Observing that $\chi(I) = I$, one has $r \in P_{I,I}$ and $-s \in Q_{I,I}$. Therefore, $r \in D^*N_C\left(x^0, v^0\right)(s)$ by definition of D^*N_C and by Proposition 3.2. ■

Corollary 3.6 shows that the conic part in the representation of the co-derivative comes into play only if strict complementarity is violated. For later purpose, we give a slightly more handy formulation of Corollary 3.6:

Corollary 3.7 *If $J = I$, then*

$$r \in D^*N_C\left(x^0, v^0\right)(s) \Longleftrightarrow s \in \ker A_I \text{ and } r \in \operatorname{im} A_I^T.$$

Here, A_I refers to the matrix whose row vectors are the a_i for $i \in I$.

4 Application to the electricity market model

In this section, we illustrate the results of the previous section by applying them to special instances of the electricity market model. We consider the EPEC (7). For the simplicity of the presentation, we restrict our considerations to so-called interior solutions. By this we mean a solution $\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)$ of (7) satisfying

$$\bar{\alpha}_i, \bar{\beta}_i > 0, \quad 0 < \bar{q}_i < \hat{q}_i, \quad -\hat{y}_i < \bar{y}_i < \hat{y}_i \quad (i = 1, \ldots, N). \tag{22}$$

Recall that $\left(\bar{\alpha}, \bar{\beta}, \bar{q}, \bar{y}\right)$ being a solution of (EPEC) implicitly entails that $(\bar{q}, \bar{y}) \in G$. The positivity of the bidding coefficients $\bar{\alpha}_i, \bar{\beta}_i$ is a very natural assumption. The remaining relations characterize a solution, where no generator and no flow of electricity reaches its simple lower and upper bounds.

4.1 Verification of the constraint qualification

As one can see from the concrete shape of F in (8), this mapping is bilinear in the couple (β, q) of variables. Thus, it fails to be polyhedral and, in order to apply the first order necessary conditions of Theorem 2.1, one first has to verify the constraint qualification of that same theorem.

Lemma 4.1 *If the incidence matrix B of the electricity network has rank m (i.e., the network is acyclic), then any interior solution to (6) satisfies the constraint qualification of Theorem 2.1.*

Proof. We ignore the equation in (10) and observe that, using the partition $v = (v_a, v_b)$, the inclusion in (10) may be written as

$$-\begin{pmatrix} 2\,[\operatorname{diag}\beta]\,v_a \\ 0 \end{pmatrix} \in D^*N_G((\bar{q},\bar{y})\,, -F\left(\bar{\alpha},\bar{\beta},\bar{q},\bar{y}\right))(v). \tag{23}$$

Now, $(\bar{q},\bar{y}) \in G$ implies that $\bar{q}+B\bar{y} \geq d$. If any inequality in this system were strict, then one could strictly decrease the cost function $c_i(q_i)$ in (4). This is because $\bar{\alpha}_i, \bar{\beta}_i > 0$ (see (22)) and so c_i is strictly increasing. Then, however, $(\bar{q},\bar{y})$ could not be a solution of (4). Consequently, $\bar{q}+B\bar{y} = d$ and so $I = \{1,\ldots,N\}$ for the set of active indices defined in Section 3 (note that the other inequalities defining G are non-binding due to assumption (22)). It follows that for some $\lambda \in \mathbb{R}^N_+$, (5) may be transformed into

$$\begin{pmatrix} \bar{\alpha}+2\left[\operatorname{diag}\bar{\beta}\right]\bar{q} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ B^T\lambda \end{pmatrix}. \tag{24}$$

By (22), comparison of the first components yields that $\lambda_i > 0$ for all $i \in \{1,\ldots,N\}$. Hence, $J = I$ for the index set introduced below (13). This allows to apply Corollary 3.7. We note that the matrix A_I occuring in this corollary coincides in our concrete setting with the matrix $-\left(I\,|B\right)$ describing the inequality system $\bar{q}+B\bar{y} \geq d$ which was actually shown to be active in each of its components. The minus-sign is due to the fact that the polyhedron C in section 3 is described by means of '$\leq$'- inequalities. Applying now Corollary 3.7 to (23) one obtains the relations

$$v_a + Bv_b = 0; \quad \begin{pmatrix} 2\left[\operatorname{diag}\bar{\beta}\right]v_a \\ 0 \end{pmatrix} = \begin{pmatrix} \mu \\ B^T\mu \end{pmatrix} \tag{25}$$

for a certain multiplier vector $\mu \in \mathbb{R}^N$. Combination of the two components in the second equation provides

$$B^T\left[\operatorname{diag}\bar{\beta}\right]Bv_b = 0.$$

Since $\bar{\beta}_i > 0$ for all $i = 1,\ldots,N$ according to (22) and B has rank m by assumption, it follows that the (m,m)- matrix $B^T\left[\operatorname{diag}\bar{\beta}\right]B$ has rank m too. Hence, $v_b = 0$ and, referring to the first equation of (25), $v_a = 0$, and so $v = 0$, as was to be shown. ∎

We do not continue here to derive the first order necessary conditions from Theorem 2.1 because it turns out that these do not uniquely identify a stationary solution. Rather a continuum of such solutions is obtained. This is consistent with a corresponding observation in [11] related to simultaneous bidding of linear and quadratic cost coefficients. We shall rather follow the idea in [11] to consider partial bidding of say linear cost coefficients in order to identify solutions. Before doing so, we generalize our setting by allowing the demands d_i in (3) to be random.

4.2 Formulation of a stochastic equilibrium problem under equilibrium constraints (SEPEC)

Since every player $i \in \{1, \dots, N\}$ does not know the demands d_j at least for $j \neq i$, but hopefully has access to historical data, it is natural to assume that d is a random vector on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose probability distribution is known (approximately). This assumption leads to a polyhedral-valued multifunction G defined on Ω with values in $\mathbb{R}^{N+m}$ given by

$$G(\omega) := \left\{ (q, y) \in \mathbb{R}^{N+m} \,\middle|\, q + By \geq d(\omega),\ 0 \leq q \leq \hat{q},\ -\hat{y} \leq y \leq \hat{y} \right\}.$$

Hence, the pair (q, y) of generation and flow has to be considered as a $\mathbb{R}^{N+m}$-valued random vector on $(\Omega, \mathcal{F}, \mathbb{P})$ and the ISO has to minimize the expected overall costs, i.e.,

$$\min_{q,y} \left\{ \mathbb{E}\left(\sum_{i=1}^{N} c_i(q_i(\omega)) \right) \,\middle|\, (q(\omega), y(\omega)) \in G(\omega),\ \mathbb{P}\text{-a.s.} \right\}. \tag{26}$$

Furthermore, the EPEC (7) now becomes the following stochastic equilibrium problem with equilibrium constraints (SEPEC)

$$\begin{aligned} \min_{\alpha_i, \beta_i, q(\cdot), y(\cdot)} \Bigg\{ \mathbb{E}\left((\gamma_i - \alpha_i)\, q_i(\omega) + (\delta_i - 2\beta_i)\, q_i^2(\omega) \right) \,\Bigg|\, 0 \in \begin{pmatrix} \alpha + 2\,[\operatorname{diag}\beta]\, q(\omega) \\ 0 \end{pmatrix} \\ + N_{G(\omega)}(q(\omega), y(\omega)),\ \mathbb{P}\text{-a.s.} \Bigg\} \quad (i = 1, \dots, N), \end{aligned} \tag{27}$$

where the pairs (α_i, β_i), $i = 1, \dots, N$, are deterministic and have to be determined before the realization of the demand, and the pairs $(q_i(\cdot), y_i(\cdot))$ $i = 1, \dots, N$, are stochastic. In the terminology of two-stage stochastic programming with recourse, the cost coefficients (α_i, β_i) are first-stage decisions, while $(q_i(\cdot), y_i(\cdot))$ are second-stage or recourse decisions.

Notice that the stochastic EPEC (27) is well defined if $G(\omega) \neq \emptyset$ holds $\mathbb{P}$-a.s. This fact is a consequence of the measurability of the set-valued mapping G (e.g., [23, Theorem 14.36]). Due to measurable selection theorems (see, e.g., [23, Corollary 14.6]) there exists a measurable function $(q(\cdot), y(\cdot)) : \Omega \to \mathbb{R}^{N+m}$ such that $(q(\omega), y(\omega)) \in G(\omega)$, $\mathbb{P}$-a.s. The expectations exist since q is bounded by $\hat{q}$.

The stochastic EPEC (27) corresponds to a coupled system of (specific) stochastic MPECs. Theoretical aspects of stochastic MPECs and their solution by sampling methods are studied in [26, 27]. Existence and stability results for solutions and numerical methods for stochastic EPECs are widely open.

4.3 Identification of M-stationary solutions for discrete random demands and partial bidding of linear coefficients

Assume that the probability distribution of d is discrete with finite support and denote by $d^{(1)}, \ldots, d^{(K)} \in \mathbb{R}^N$ the K different scenarios of d. The scenarios induce K different polyhedra of scenario-dependent generation and transmission constraints

$$G_k := \left\{(q, y) \in \mathbb{R}^{N+m} \,\middle|\, q + By \geq d^{(k)},\ 0 \leq q \leq \hat{q},\ -\hat{y} \leq y \leq \hat{y}\right\} \quad (k = 1, \ldots, K).$$

According to the remarks at the end of Section 4.1, we suppose now the quadratic bid coefficients to be known, hence, $\beta = \delta$, and only the linear bid coefficients to be subject of optimization. The generalized equation (5) now has to be established for each scenario k as follows:

$$0 \in \begin{pmatrix} \alpha + 2\,[\operatorname{diag}\delta]\, q^{(k)} \\ 0 \end{pmatrix} + N_{G_k}(q^{(k)}, y^{(k)}) \quad k = 1, \ldots, K. \tag{28}$$

Accordingly, generator i's profit under scenario k equals

$$(\alpha_i - \gamma_i)\, q_i^{(k)*} + \delta_i \left(q_i^{(k)*}\right)^2,$$

where $q^{(k)*}$ is a solution of (28). Then, in order that every generator maximizes its expected profit, the underlying SEPEC becomes

$$\min \left\{ f_i\left(\alpha_i, q, y\right) \,\middle|\, 0 \in F^{(k)}(\alpha, q, y) + N_{G_k}(q^{(k)}, y^{(k)}) \quad (k = 1, \ldots, K) \right\} \\ (i = 1, \ldots, N), \qquad (SEPEC)$$

where $q = \left(q^{(1)}, \ldots, q^{(K)}\right)$, $y = \left(y^{(1)}, \ldots, y^{(K)}\right)$ and

$$\begin{aligned} f_i\left(\alpha_i, q, y\right) &= \sum_{k=1}^{K} p_k \left[(\gamma_i - \alpha_i)\, q_i^{(k)} - \delta_i \left(q_i^{(k)}\right)^2 \right] \quad (i = 1, \ldots, N), \\ F^{(k)}\left(\alpha, q, y\right) &= \begin{pmatrix} \alpha + 2\,[\operatorname{diag}\delta]\, q^{(k)} \\ 0 \end{pmatrix} \quad (k = 1, \ldots, K). \end{aligned}$$

Here, the p_k are the probabilities for the demand scenarios $d^{(k)}$, so in particular they fulfill the relations

$$\sum_{k=1}^{K} p_k = 1, \quad p_k \geq 0 \quad (k = 1, \ldots, K).$$

In order to apply Theorem 2.1, we rewrite (SEPEC) as a usual EPEC. To this aim we put

$$F := \left(F^{(1)}, \ldots, F^{(K)}\right), \quad G := G_1 \times \cdots \times G_K.$$

Owing to the calculus rule

$$N_G\left(q, y\right) = N_{G_1}(q^{(1)}, y^{(1)}) \times \cdots \times N_{G_K}(q^{(K)}, y^{(K)}),$$

(SEPEC) boils down to (EPEC) as presented in Section 2. Since F is a linear mapping, the multifunction (9) is polyhedral and we may directly apply the necessary optimality conditions of Theorem 2.1 without checking the constraint qualification.

As in Section 4.1, we shall be interested in so-called interior solutions for the ease of presentations. Owing to the scenario character of parts of the solution, we have to make this concept more precise: A solution $(\bar{\alpha}, \bar{q}, \bar{y})$ of (7) with the data specified above is called an interior solution, if it satisfies

$$\bar{\alpha}_i > 0, \quad 0 < \bar{q}_i^{(k)} < \hat{q}_i, \quad -\hat{y}_i < \bar{y}_i^{(k)} < \hat{y}_i \quad (i = 1, \ldots, N; k = 1, \ldots, K)\,. \tag{29}$$

Recalling, that partial derivative just with respect to α_i rather than with respect to (α_i, β_i) have to be considered now, we deal with

$$\begin{aligned} \nabla_{\alpha_i} f_i\left(\alpha_i, q, y\right) &= -\sum_{k=1}^{K} p_k q_i^{(k)} \\ \left[\nabla_{\alpha_i} F\left(\alpha, q, y\right)\right]^T &= \left(\left(e_i^T, 0\right) |\ldots| \left(e_i^T, 0\right)\right), \end{aligned}$$

where e_i denotes the i-th standard unit vector in $\mathbb{R}^N$. Then, writing the i-th multiplier in the necessary optimality conditions as

$$\bar{v}_i = \left(\bar{v}_i^{(1)}, \ldots, \bar{v}_i^{(K)}\right),$$

the first equation (11) becomes

$$\sum_{k=1}^{K} p_k \bar{q}_i^{(k)} = \sum_{k=1}^{K} \bar{v}_{ii}^{(k)}. \tag{30}$$

Next, repeating (scenario-wise) the same argumentation as the one leading to (24), and taking into account that $\beta = \delta$, one verifies the existence of $\lambda^{(k)} \in \mathbb{R}^N_+$, such that

$$\begin{pmatrix} \bar{\alpha} + 2\left[\operatorname{diag}\delta\right] \bar{q}^{(k)} \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda^{(k)} \\ B^T \lambda^{(k)} \end{pmatrix} \quad (k = 1, \ldots, K).$$

This may be condensed to the relations

$$B^T(\bar{\alpha} + 2\left[\operatorname{diag}\delta\right] \bar{q}^{(k)}) = 0 \quad (k = 1, \ldots, K). \tag{31}$$

When describing the polyhedron G introduced above as an inequality system of the type $Ax \leq b$ as required in Section 3, one would have to put

$$A := \begin{pmatrix} \tilde{A} & & 0 \\ & \ddots & \\ 0 & & \tilde{A} \end{pmatrix}, \quad \tilde{A} := \begin{pmatrix} -I & -B \\ -I & 0 \\ I & 0 \\ 0 & -I \\ 0 & I \end{pmatrix},$$

$$x := \left(q^{(1)}, y^{(1)}, \cdots, q^{(K)}, y^{(K)}\right)^T, \quad b := \left(-d^{(1)}, 0, \hat{q}, -\hat{y}, \hat{y}, \cdots, -d^{(K)}, 0, \hat{q}, -\hat{y}, \hat{y}\right)^T$$

On the other hand, looking for interior solutions according to (29), only the inequalities of the type $q^{(k)} + By^{(k)} \geq d^{(k)}$ are binding (compare discussion in the beginning of the proof of Lemma 4.1). Hence,

$$q^{(k)} + By^{(k)} = d^{(k)} \quad (k = 1, \ldots, K) \tag{32}$$

and the matrix A_I introduced in Corollary 3.7 has the shape

$$A_I = \begin{pmatrix} (-I \,|-B\,) & & 0 \\ & \ddots & \\ 0 & & (-I \,|-B\,) \end{pmatrix}.$$

Then, with the partition $\bar{v}_i^{(k)} = ([\bar{v}_i^{(k)}]_a, [\bar{v}_i^{(k)}]_b)$, the first statement of Corollary 3.7 allows to extract the following two conditions from the inclusion (12):

$$[\bar{v}_i^{(k)}]_a + B[\bar{v}_i^{(k)}]_b = 0 \quad (i = 1, \ldots, N;\, k = 1, \ldots, K). \tag{33}$$

Moreover,

$$\begin{aligned} \nabla_y f_i &= 0 \\ \nabla_q f_i &= (\nabla_{q^{(1)}} f_i, \ldots, \nabla_{q^{(K)}} f_i) \quad (i = 1, \ldots, N), \quad \text{where} \\ \nabla_{q^{(k)}} f_i(\alpha_i, q, y) &= (0, \ldots, 0, p_k[\gamma_i - \alpha_i - 2\delta_i q_i^{(k)}], 0, \ldots, 0) \end{aligned}$$

and

$$\begin{aligned} \nabla_y F &= 0 \\ \nabla_q F(\alpha, q, y)^T \bar{v}_i &= \begin{pmatrix} 2[\operatorname{diag}\delta][\bar{v}_i^{(1)}]_a \\ \cdots \\ 2[\operatorname{diag}\delta][\bar{v}_i^{(K)}]_a \end{pmatrix} \quad (i = 1, \ldots, N). \end{aligned}$$

Thus, the second statement of Corollary 3.7 together with the inclusion (12) yields the existence of multipliers $\mu^{(k)} \in \mathbb{R}^n$ such that

$$\begin{aligned} \begin{pmatrix} w_i^{(k)} \\ 0 \end{pmatrix} &= \begin{pmatrix} \mu^{(k)} \\ B^T \mu^{(k)} \end{pmatrix} \quad (k = 1, \ldots, K; i = 1, \ldots, N), \quad \text{where} \\ w_i^{(k)} := & \quad (2\delta_1 \bar{v}_{i1}^{(k)}, \ldots, 2\delta_{i-1} \bar{v}_{i,i-1}^{(k)}, 2\delta_i \bar{v}_{ii}^{(k)} + p_k[\gamma_i - \bar{\alpha}_i - 2\delta_i \bar{q}_i^{(k)}], \\ & \quad 2\delta_{i+1} \bar{v}_{i,i+1}^{(k)}, \ldots, 2\delta_N \bar{v}_{iN}^{(k)})^T. \end{aligned}$$

In brief,

$$B^T w_i^{(k)} = 0 \quad (k = 1, \ldots, K; i = 1, \ldots, N) \tag{34}$$

Summarizing, M-stationary solutions of (SEPEC) are characterized by the relations (30), (31), (32), (33) and (34).

4.4 Explicit calculation of M-stationary solutions for a small example

Finally, we want to illustrate the results of the previous section by explicitly calculating the solution of (SEPEC) for the smallest meaningful example, namely a network consisting of $N = 2$ nodes which are linked by one single arc ($m = 1$). In this case, the incidence matrix simply becomes

$$B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

First, (30) may be shortly written as

$$\mathbb{E}\bar{q}_i = \sum_{k=1}^{K} \bar{v}_{ii}^{(k)} \quad (i = 1, 2), \tag{35}$$

where 'E' refers to the expected value. With the concrete shape of B, (31) takes the form

$$\bar{\alpha}_1 + 2\delta_1 \bar{q}_1^{(k)} = \bar{\alpha}_2 + 2\delta_2 \bar{q}_2^{(k)} \quad (k = 1, \ldots, K). \tag{36}$$

Summing up all these equations upon multiplying them by the probabilities p_k, one arrives at

$$\bar{\alpha}_1 + 2\delta_1 \mathbb{E}\bar{q}_1 = \bar{\alpha}_2 + 2\delta_2 \mathbb{E}\bar{q}_2. \tag{37}$$

Next, we derive from (34) the equations

$$\left.\begin{array}{l} 2\delta_1 \bar{v}_{11}^{(k)} + p_k[\gamma_1 - \bar{\alpha}_1 - 2\delta_1 \bar{q}_1^{(k)}] = 2\delta_2 \bar{v}_{12}^{(k)} \\ 2\delta_2 \bar{v}_{22}^{(k)} + p_k[\gamma_2 - \bar{\alpha}_2 - 2\delta_2 \bar{q}_2^{(k)}] = 2\delta_1 \bar{v}_{21}^{(k)} \end{array}\right\} \quad (k = 1, \ldots, K). \tag{38}$$

Summing up over k the upper equations, we get

$$2\delta_1 \sum_{k=1}^{K} \bar{v}_{11}^{(k)} + \gamma_1 - \bar{\alpha}_1 - 2\delta_1 \mathbb{E}\bar{q}_1 = 2\delta_2 \sum_{k=1}^{K} \bar{v}_{12}^{(k)}.$$

Taking into account (35), this reduces to

$$\gamma_1 - \bar{\alpha}_1 = 2\delta_2 \sum_{k=1}^{K} \bar{v}_{12}^{(k)}. \tag{39}$$

Furthermore, (33) yields

$$\bar{v}_{11}^{(k)} = -\bar{v}_{12}^{(k)}, \quad \bar{v}_{21}^{(k)} = -\bar{v}_{22}^{(k)} \quad (k = 1, \ldots, K). \tag{40}$$

Combining the first of these relations with (39) and (35), we obtain

$$\gamma_1 - \bar{\alpha}_1 + 2\delta_2 \mathbb{E}\bar{q}_1 = 0. \tag{41}$$

Similarly, the corresponding second relations in (38) and (40) allow to derive that

$$\gamma_2 - \bar{\alpha}_2 + 2\delta_1 \mathbb{E}\bar{q}_2 = 0. \tag{42}$$

Finally, reading the components of (32) with the concrete shape of B gives

$$\bar{q}_1^{(k)} + \bar{y}^{(k)} = d_1^{(k)}; \quad \bar{q}_2^{(k)} - \bar{y}^{(k)} = d_2^{(k)} \quad (k = 1, \ldots, K) \tag{43}$$

Adding both equations leads to

$$\bar{q}_1^{(k)} + \bar{q}_2^{(k)} = d_1^{(k)} + d_2^{(k)} \quad (k = 1, \ldots, K). \tag{44}$$

Summation over k entails that $\mathbb{E}\bar{q}_1 + \mathbb{E}\bar{q}_2 = \mathbb{E}d_1 + \mathbb{E}d_2$. Now, this last equation along with (37), (41) and (42) constitutes a system of four linear equations in the four unknowns $\bar{\alpha}_1$, $\bar{\alpha}_2$, $\mathbb{E}\bar{q}_1$ and $\mathbb{E}\bar{q}_2$, which is easily resolved for its solution

$$\begin{aligned}
\bar{\alpha}_1 &= \gamma_1 + \delta_2 \left(\mathbb{E}d_1 + \mathbb{E}d_2 + \frac{\gamma_2 - \gamma_1}{2(\delta_1 + \delta_2)} \right) \\
\bar{\alpha}_2 &= \gamma_2 + \delta_1 \left(\mathbb{E}d_1 + \mathbb{E}d_2 + \frac{\gamma_1 - \gamma_2}{2(\delta_1 + \delta_2)} \right) \\
\mathbb{E}\bar{q}_1 &= \frac{1}{2} (\mathbb{E}d_1 + \mathbb{E}d_2) + \frac{\gamma_2 - \gamma_1}{4(\delta_1 + \delta_2)} \\
\mathbb{E}\bar{q}_2 &= \frac{1}{2} (\mathbb{E}d_1 + \mathbb{E}d_2) + \frac{\gamma_1 - \gamma_2}{4(\delta_1 + \delta_2)}.
\end{aligned}$$

With these $\bar{\alpha}_1$ and $\bar{\alpha}_2$ one may combine (44) and (36) in order to identify the scenario-dependent amounts of electricity generation of both competitors:

$$\begin{aligned}
\bar{q}_1^{(k)} &= \frac{\frac{1}{2}(\gamma_2 - \gamma_1) + (\delta_1 - \delta_2)(\mathbb{E}d_1 + \mathbb{E}d_2) + 2\delta_2 \left(d_1^{(k)} + d_2^{(k)} \right)}{2(\delta_1 + \delta_2)} \quad (k = 1, \ldots, K) \\
\bar{q}_2^{(k)} &= \frac{\frac{1}{2}(\gamma_1 - \gamma_2) + (\delta_2 - \delta_1)(\mathbb{E}d_1 + \mathbb{E}d_2) + 2\delta_1 \left(d_1^{(k)} + d_2^{(k)} \right)}{2(\delta_1 + \delta_2)} \quad (k = 1, \ldots, K).
\end{aligned}$$

Next, using either of the two equations in (43), we may resolve for the scenario-dependent amount of electricity sent from node 2 to node 1:

$$\bar{y}^{(k)} = \frac{1}{2}(\gamma_1 - \gamma_2) + (\delta_2 - \delta_1)(\mathbb{E}d_1 + \mathbb{E}d_2) + 2\delta_1 d_1^{(k)} - 2\delta_2 d_2^{(k)} \quad (k = 1, \ldots, K).$$

The expected value of this flow calculates as

$$\mathbb{E}\bar{y} = \frac{1}{2}(\gamma_1 - \gamma_2) + (\delta_1 + \delta_2)(\mathbb{E}d_1 - \mathbb{E}d_2).$$

Finally, we determine the expected profits $\mathbb{E}\pi_i$ of both competing generators:

$$\begin{aligned} \mathbb{E}\pi_1 &= \sum_{k=1}^{K} p_k \left[(\bar{\alpha}_1 - \gamma_1)\, \bar{q}_1^{(k)} + \delta_1 \left(\bar{q}_1^{(k)} \right)^2 \right] = (\bar{\alpha}_1 - \gamma_1)\, \mathbb{E}\bar{q}_1 + \delta_1 \mathbb{E}\, (\bar{q}_1)^2 \\ \mathbb{E}\pi_2 &= (\bar{\alpha}_2 - \gamma_2)\, \mathbb{E}\bar{q}_2 + \delta_2 \mathbb{E}\, (\bar{q}_2)^2 . \end{aligned}$$

References

[1] B. Blaesig: *Risikomanagement in der Stromerzeugungs- und Handelsplanung*, Aachener Beiträge zur Energieversorgung, Band 113, Aachen 2007.

[2] A. L. Dontchev and R. T. Rockafellar, Characterizations of strong regularity for variational inequalities over polyhedral convex sets, *SIAM J. Optim.* 7 (1996), 1087-1105.

[3] A. Eichhorn and W. Römisch, Mean-risk optimization models for electricity portfolio management, *Proceedings of the 9th International Conference on Probabilistic Methods Applied to Power Systems* (PMAPS 2006), Stockholm, 2006.

[4] J. F. Escobar and A. Jofre, Oligopolistic competition in electrictiy spot markets, Dec. 2005, available at `http://ssrn.com/abstract=878762`.

[5] F. Facchinei and J.-S. Pang: Finite-dimensional Variational Inequalities and Complementarity Problems, Vol. I and II, Springer, New York, 2003.

[6] R. Fletcher, S. Leyffer, D. Ralph and S. Scholtes: Local convergence of SQP methods for mathematical programs with equilibrium constraints. *SIAM J. Optim.* 17 (2006), 259-286.

[7] R. Garcia-Bertrand, A. J. Conejo, S. Gabriel: Electricity market near-equilibrium under locational marginal pricing and minimum profit conditions, *European Journal of Operational Research* 174 (2006), 457–479.

[8] B. F. Hobbs and J.-S. Pang: Nash-Cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints, *Operations Research* 55 (2007), 113–127.

[9] B. F. Hobbs, C. Metzler and J.-S. Pang: Strategic gaming analysis for electric power networks: An MPEC approach, IEEE Power Engineering Transactions 15 (2000), 638-645.

[10] X. Hu and D. Ralph, Using EPECs to model bilevel games in restructured electricity markets with locational prices, Optimization Online 2006 (`www.optimization-online.org`).

[11] X. Hu, D. Ralph, E. K. Ralph, P. Bardsley and M. C. Ferris, Electricity generation with looped transmission networks: Bidding to an ISO, Research Paper No. 2004/16, Judge Institute of Management, Cambridge University, 2004.

[12] M. Kočvara and J. V. Outrata: Optimization problems with equilibrium constraints and their numerical solution, *Math. Programming* B 101 (2004), 119–150.

[13] S. Leyffer and T. S. Munson. Solving multi-leader-follower games. Preprint ANL/MCS-P1243-0405, Argonne National Laboratory, Mathematics and Computer Science Division, April 2005.

[14] Z.-Q. Luo, J.-S. Pang and D. Ralph: *Mathematical Programs with Equilibrium Constraints*, Cambridge University Press, Cambridge, 1996.

[15] B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation. Vol. 1: Basic Theory, Vol. 2: Applications*, Springer, Berlin, 2006.

[16] B. S. Mordukhovich and J. V. Outrata, Coderivative analysis of quasi-variational inequalities with applications to stability and optimization, *SIAM J. Optim.* 18 (2007), 389-412.

[17] J. V. Outrata, A note on a class of equilibrium problems with equilibrium constraints, *Kybernetica*, 40 (2004), 585-594.

[18] J. V. Outrata, On constrained qualifications for mathematical programs with mixed complementarity constraints, In (M.C. Ferris et al. eds.): Complementarity: Applications, Algorithms and Extensions, Kluwer, Dordrecht, 2001, 253-272.

[19] J. V. Outrata, M. Kočvara and J. Zowe: *Nonsmooth Approach to Optimization Problems with Equilibrium Constraints*, Kluwer, Dordrecht, 1998.

[20] J.-S. Pang and M. Fukushima: Quasi-variational inequalities, generalized Nash equilibria, and multi-leader-follower games, Computational Management Science 1 (2005), 21–56.

[21] M. V. Pereira, S. Granville, M. H. C. Fampa, R. Dix and L. A. Barroso: Strategic bidding under uncertainty: A binary expansion approach, *IEEE Transactions on Power Systems* 20 (2005), 180–188.

[22] D. Ralph and Y. Smeers, EPECs as models for electricity markets, Power Systems Conference and Exposition (PSCE), Atlanta, 2006.

[23] R. T. Rockafellar and R. J-B Wets: *Variational Analysis*, Springer, Berlin 1998.

[24] A. Ruszczyński and A. Shapiro (Eds.): *Stochastic Programming*, Handbooks in Operations Research and Management Science, Volume 10, Elsevier, Amsterdam 2003.

[25] H. Scheel and S. Scholtes: Mathematical programs with equilibrium constraints: Stationarity, optimality and sensitivity, *Math. Oper. Res.* 25 (2000), 1–22.

[26] A. Shapiro: Stochastic programming with equilibrium constraints, *Journal Optimization Theory and Applications* 128 (2006), 223–243.

[27] A. Shapiro and H. Xu: Stochastic mathematical programs with equilibrium constraints, modeling and sample average approximation, Optimization Online 2005 (`www.optimization-online.org`).

[28] Y. Smeers: How well can one measure market power in restructured electricity systems? CORE Discussion Paper 2005/50, Center for Operations Research and Econometrics (CORE), Louvain, 2005.